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Supply Chain Model for Three-Echelon Supply Chain for Perishable Products

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Abstract

This study comprehensively explores three distinct supply chain models designed explicitly for managing perishable products, including categories such as fruits, fashion goods, and vegetables. Our research focuses on a three-tier supply structure, illustrating manufacturing and consumer delivery dynamics. In this framework, the process begins with the manufacturer, who produces these perishable items. Once the products are ready, they are supplied to an intermediary known as the supplier. This supplier plays a crucial role in the supply chain, acting as a bridge between the manufacturer and the retailer, thus establishing a structured echelon supply chain model. Upon getting the goods from the supplier, the retailer is responsible for marketing and selling these products directly to customers. Within this context, the retailer must consider various factors, particularly the seasonal demand patterns that influence customer purchasing behavior and the unpredictable lead times associated with deliveries from the supplier. These elements are critical, as they can significantly impact inventory levels and the freshness of the perishable products.

Keyword: supply chain management, deterioration, price-dependent demand, backordering.

I. INTRODUCTION

Supply chain management is a hot topic in business today. The idea is to manage the entire flow of information, materials, and services from the raw materials stage to the finished goods stage. Supplies are a necessity for any production unit. The essential part of any supply chain model is inventory management; in this study, we have considered an inventory model of deteriorating items for the three echelon supply chains.

No business works in isolation. Each acts as a customer (when buying materials from suppliers) and a supplier (when delivering to customers). A wholesaler, for example, acts as a customer when purchasing goods from manufacturers and then as a supplier when selling goods to retail shops. Product moves through organizations and operations as they travel between original suppliers and final customers. Before buying it, milk moves through a farm, tanker collection, dairy, bottling plant, distributor, and supermarket. A toothbrush starts its journey with a company extracting crude oil. Then it passes through pipelines, refineries, chemical works, plastics companies,

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manufacturers, importers, wholesalers, and retailers before finishing up in your bathroom. The series of activities and organizations form the product's supply chain. The function that has overall responsibility for moving through the supply chain is logistics or supply chain management

In most of the inventory models considered in the literature, demanded items are directly delivered from the stock (if available). The demand occurring during the stock-out period is either lost or satisfied only after ordering items arrive, which is the case of backlogging. We have assumed that demand during the stock-out period is partially met. In some inventory systems, the length of the waiting time for the next replenishment becomes the main factor for fashionable goods. If the waiting time is longer, the backlogging rate is lower. So the backlogging rate depends on the waiting time for the next replenishment.

The study proposes a three echelon model for the supply chain and tires to come out with the optimal solution which helps all the stakeholders in getting maximum benefits.

1.1. Background Literature

Barbosa and Fiedman (1978) framed an inventory model with a deterministic demand pattern and shortage. Petruzzi and Dada (1999) developed an inventory problem to determine the order quantity and sale price. Ghare and Schrader (1963) developed an EOQ model by assuming a constant rate of deterioration. Mandal and Phaujdar (1989) developed a production inventory model for deterioration. Aggarwal and Jaggi (1995) and Shinn and Hwang (2003) worked on Goyal's (1985) model to formulate the deterministic inventory model with a constant deterioration rate. After Aggarwal and Jaggi (1995), Jamal et al. (1997) extended their model to allow for shortages and make it more applicable in the real world. A model in the field of deteriorating items with time-varying demand and shortages has recently been developed by Chang and Dye (1999). The backlogging rate is inversely proportional to the waiting time for the next replenishment. Chung and Lin (2001) extended the inventory replenishment model of Chung to the situation where shortages are allowed in each replenishment cycle. The model starts with inventory and ends with shortages. This study considers three echelon supply chains. After production, the manufacturer supplies the deteriorating products to the supplier, and then the retailer receives the product from the supplier for sale to the customers. The retailer considers the customer's seasonal pattern demand and supplier's uncertain lead time. In this model, the product expiration date is also considered.

II. LITERATURE REVIEW

Hsu et al. (2006) established a deteriorating inventory model for season pattern demand with an expiration date. The product expiration date indicates the latest time that product may be used (not the end of the product life cycle time) and the customer 's demand always decreases as the product is nearer to the expiration date. Wu et al. (2006) considered a problem to determine the optimal replenishment policy for non-instantaneous deteriorating products with stock-dependent time. Dye et al. (2007) proposed a deterministic inventory model for deteriorating items with capacity constraints and time proportional backlogging rate.

Shah and Shukla (2009) formulated a deterministic inventory model in which product is subject to constant deterioration and shortages are allowed. Min et al. (2010) had developed a lot-sizing model for deteriorating items with a current-stock-dependent demand and delay in payments. Mahata (2012) had examined the optimum retailer's replenishment decisions for deteriorating items under two levels of trade credit policy to reflect supply chain management within the economic production quantity (EPQ) framework. Liao et al. (2013) had developed recently a two-warehouse inventory model for perishable products when the supplier offers the retailer a delay period and in turn, the retailer provides a delay period to their customers. Dash et al. (2014), formulated an inventory model for deteriorating items with exponential decreasing demand and time-dependent holding cost. Singh (2017) obtained an optimal ordering policy for deteriorating goods having constant demand. Saha and Sen (2019) developed an inventory model for deteriorating items with time and price-dependent demand and shortages under the effect of inflation San-José et al. (2021) formulated the pricing policy for optimal cost considering the inventory model with time-and-price-dependent demand and shortages.

This study considers three echelon supply chains. After production, the manufacturer supplies the deteriorating products to the supplier, and then the retailer receives the product from the supplier for sale to the customers. The retailer considers the customer's seasonal pattern demand and supplier's uncertain lead time. In this model, the product expiration date is also considered.

This study aims to derive the retailer's optimal replenishment cycle, shortage period, economic order quantity, and the managing cost of the supplier to maximize the expected unit time profit.

III. RESEARCH METHODOLOGY

In this study, we have investigated a supply chain inventory model for deteriorating items in which demand depends on time and price. The shortages allowed and partially backlogged, and it is assumed that backlogging rate varies inversely as the waiting time. Demand is seasonal, along with the expiration rate. The decision variables in this article are the length of the replenishment cycle, shortage period, order quantity, and the managing cost. Using the mathematical modeling the differential equation based model is established using the following assumptions

3.1. Assumptions

1) Selling price per unit p and backorder price is such that $pb = \lambda p \rightarrow 0<\lambda<1$
2) The warehouse has unlimited capacity.
3) Supplier's delivery does not go beyond the second season.
4) There is no replacement of deteriorated items during a given cycle.
5) The product's seasonal pattern demand is the function of price and season such that

$$dj(t,p) = \begin{cases} \dfrac{Aw(j)}{p^{\beta}} & j = 1, 2, \ldots \\ 0 & j > n \end{cases}$$

and w(j) is given by:

$$w(j) = \frac{N-j+1}{N}$$

Here α and β are known constant with $\alpha, \beta > 0$.

6) Demand during the stock out period is partially lost and fraction of customers' backordered is given by

$$\theta(\eta) = 1 - \eta/T \rightarrow 0 \le \eta < T$$

Where: η is the waiting time.

7) Shortage time is less than the length of the seasonal interval.
8) The replenishment cycle for the manufacturer and the retailer is set at NT.

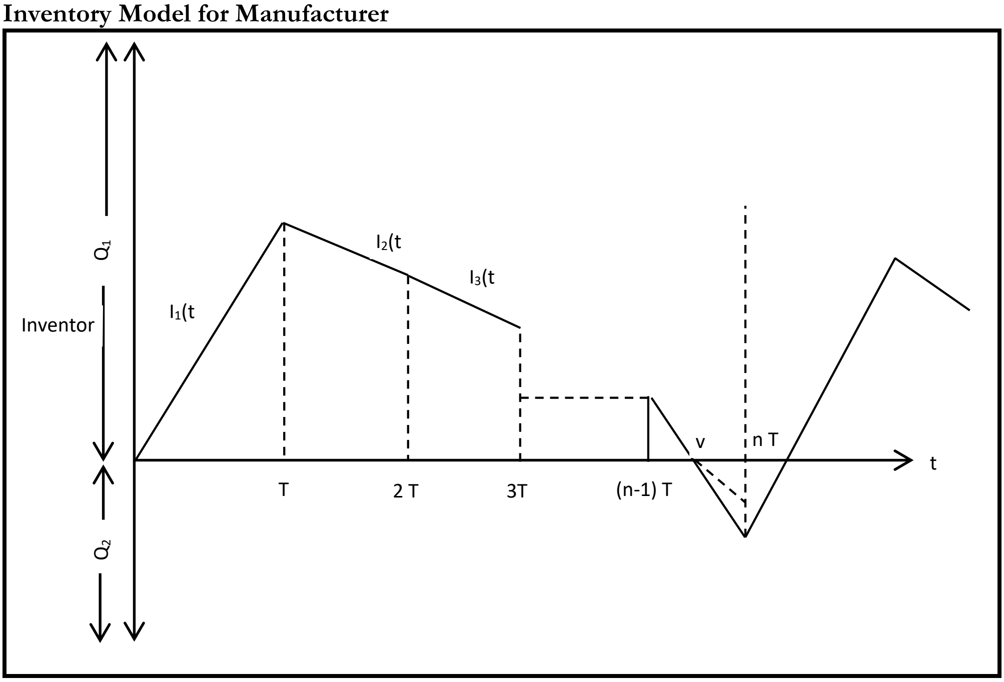
3.2. Notations used in the Model

Notation	Information
T	Length of the seasonal interval
N	A discrete number NT denotes the expiration date of the product
v	A critical time at which inventory level reaches zero in the last season
Q_m	Manufacturer's production quantity
Q_{m1}	Manufacturer's sales amount without backordering over the replenishment cycle
Q_{m2}	Manufacturer's back-ordered quantity at the end of the replenishment cycle
Q_R	Retailer's order quantity at each replenishment cycle
Q_{R1}	Retailer's sales amount without backordering over the replenishment cycle
Q_{R2}	Retailer's back ordered quantity at the end of the replenishment cycle
ξ	Managing cost
p_r	Retailer's selling price per unit
p_m	Manufacturer's selling price per unit
P_b	Selling price per unit when shortage occur
$\vartheta(\eta)$	Fraction of customer's back-ordered during the stock out period
h	Inventory holding cost per unit time
C_m	Production cost per unit
C	Retailer's wholesale purchase price per unit
C_{mo}	Manufacturer's ordering cost
C_{Ro}	Retailer's ordering cost
r	Penalty cost per unit of a lost sale including loss of profit
y	Supplier's lead time
f(y)	Probability density function of supplier's lead time

In this paper deterioration rate= $\mathbf{a+bt+ct^2}$

Figure 1

Inventory Model for Manufacturer



3.3. Manufacturer's Model

The differential equation of the system is given by

$$\frac{dI_1(t)}{dt} = P-(a+bt+ct^2)+\frac{\alpha w(1)}{p^\beta} \quad \ldots\ldots \quad 1$$

Where: $0 \le t \le T$

The solution of Equation 1 is given by

$$I_1(t).e^{at+\frac{br^2}{2}+\frac{ct^3}{3}} = \left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right)+C_1 \quad \ldots\ldots \quad 2$$

Now using initial condition $I_1(0)= 0$

$$I_1(0) = \left(P-\frac{\alpha w(1)}{p^\beta}\right).0 + C_1$$

Where: $C_1= 0$

Put this value of C_1 in Equation 2

$$I_1(t) = \left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right)+e^{-at-\frac{br^2}{2}-\frac{ct^3}{3}} \quad \ldots\ldots \quad 3$$

Now let $I_2(t)$ be the inventory level for the manufacturer during 2nd season:

$$\frac{dI_2(t)}{dt}+(a+bt+ct^2)I_2(t) = -\frac{\alpha w(2)}{p^\beta} \quad \ldots\ldots \quad 4$$

The solution of Equation 2 will be

$$I_2(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \int\left(-\frac{\alpha w(2)}{p^\beta}\right).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}}.dt + C_2$$

$$I_2(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \left(-\frac{\alpha w(2)}{p^\beta}\right)\int\left(1+at+\frac{bt^2}{2}+\frac{ct^3}{3}+\ldots\ldots\right).dt + C_2$$

Omitting the higher powers of a, b and c:

$$I_2(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \left(-\frac{\alpha w(2)}{p^\beta}\right)\left[t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right]+C_2 \quad \ldots\ldots \quad 5$$

Now, using initial condition $I_1(T) = I_2(0)$

$$I_1(T) = \left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-aT-\frac{bT^2}{2}-\frac{cT^3}{3}}$$

$$I_2(0) = \left(-\frac{\alpha w(2)}{p^\beta}\right).0 + C_2$$

$$\mathbf{I_2(0)= C_2}$$

$$C_2 = \left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-aT-\frac{bT^2}{2}-\frac{cT^3}{3}}$$

Put this value of C_2 in Equation 5

$$\int I_2(t) = \left(-\frac{\alpha w(2)}{p^\beta}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right).e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}}$$
$$+\left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a(T+t)-\frac{b}{2}(T^2+t^2)-\frac{c}{3}(T^3+t^3)} \quad \ldots\ldots \quad 6$$

If $I_3(t)$ be the inventory level during third season

$$\frac{dI_3(t)}{dt} = -(a+bt+ct^2)I_3(t) - \frac{\alpha w(3)}{p^\beta} \quad \text{.......} \quad 7$$

Where: $0 \leq t \leq T$

Solution of Equation 7 will be

$$I_3(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \left(-\frac{\alpha w(3)}{p^\beta}\right)\int\left(1+at+\frac{bt^2}{2}+\frac{ct^3}{3}+.........\right).dt + C_3$$

Omitting the higher powers of a, b and c

$$I_3(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \left(-\frac{\alpha w(3)}{p^\beta}\right)\left[t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right]+C_3 \quad \text{.......} \quad 8$$

Now using initial condition

$I_3(0)= I_2(T)$

$I_3(0)= C_3$

$$I_2(T) = \left(-\frac{\alpha w(2)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-aT-\frac{bT^2}{2}-\frac{cT^3}{3}}$$
$$+\left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a.2T-\frac{b}{2}2T^2-\frac{c}{3}.2T^3}$$

$$C_3 = \left(-\frac{\alpha w(2)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-aT-\frac{bT^2}{2}-\frac{cT^3}{3}}$$
$$+\left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a.2T-\frac{b}{2}2T^2-\frac{c}{3}.2T^3}$$

Put this value of C_3 in Equation 8, we get

$$I_3(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \left(-\frac{\alpha w(3)}{p^\beta}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right)$$
$$+\left(-\frac{\alpha w(2)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-aT-\frac{bT^2}{2}-\frac{cT^3}{3}}$$
$$+\left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a.2T-\frac{b}{2}2T^2-\frac{c}{3}.2T^3}$$

$$I_3(t) = \left(-\frac{\alpha w(3)}{p^\beta}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right).e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}}$$
$$+\left(-\frac{\alpha w(2)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a(T+t)-\frac{b}{2}(T^2+t^2)-\frac{c}{3}(T^3+t^3)}$$
$$+\left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a(2T+t)-\frac{b}{2}(2T^2+t^2)-\frac{c}{3}(2T^3+t^3)} \quad \text{.......} \quad 9$$

If $I_4(T)$ be the inventory level during 4th season

$$\frac{dI_4(t)}{dt} = -(a+bt+ct^2)I_4(t) - \frac{\alpha w(4)}{p^\beta} \quad \text{.......} \quad 10$$

After solving and omitting the higher powers of a, b and c

$$I_4(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \left(-\frac{\alpha w(4)}{p^\beta}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right)+C_4 \quad \text{.......} \quad 11$$

Now using initial condition $I_4(0)= I_3(T)$

$I_4(0)= C_4$

$$C_4=\left(-\frac{\alpha w(3)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-aT-\frac{bT^2}{2}-\frac{cT^3}{3}}+\left(-\frac{\alpha w(2)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-2aT-\frac{b}{2}.2T^2-\frac{c}{3}.2T^3}+\left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-3aT-\frac{b}{2}.3T^2-\frac{c}{3}.3T^3}$$

Put this value of C_4 in Equation 11

$$\mathbf{I_4(t)=\left(-\frac{\alpha w(4)}{p^\beta}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right).e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}}+\left(-\frac{\alpha w(3)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a(T+t)-\frac{b}{2}(T^2+t^2)-\frac{c}{3}(T^3+t^3)}+\left(-\frac{\alpha w(2)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a(2T+t)-\frac{b}{2}(2T^2+t^2)-\frac{c}{3}(2T^3+t^3)}+\left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a(3T+t)-\frac{b}{2}(3T^2+t^2)-\frac{c}{3}(3T^3+t^3)}} \quad \text{.........} \quad \mathbf{12}$$

Similarly the inventory level during the jt[h] season will be

$$I_j(t)=\left(-\frac{\alpha w(j)}{p^\beta}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right).e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}}+\sum_{\substack{\ell=j-1\\ \&m=\ell-j}}^{2}\left(-\frac{\alpha w(\ell)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{a(mT-t)+\frac{b}{2}(mT^2-t^2)+\frac{c}{3}(mT^3-t^3)}+\left(P-\frac{\alpha w(1)}{p^\beta}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a[(j-1)T+t]-\frac{b}{2}[(j-1)T^2+t^2]-\frac{c}{3}[(j-1)T^3-t^3]} \quad \mathbf{13}$$

Now, the unit time profit for the manufacturer

$F_m(n, v)= \frac{1}{nT}$ [Sales Revenue-Production Cost-Deteriorated Cost -Inventory Holding Cost-Managing Cost-Shortage Cost]

$F_m(n, v)= \frac{1}{nT}\left[R(n, v)-C_m-C_d-H(n, v)-\xi_m-S.C._m\right]$ **14**

Where: $R(n, v)= pQm_1+\lambda pQm_2$

$$\mathbf{Q_{m_1}=\int_0^T d_1(t,p)dt+\sum_{j=2}^{n-1}\int_0^T d_j(t,p)dt+\int_0^v d_n(t,p)dt} \quad \text{..................................} \quad \mathbf{15}$$

$$\mathbf{Q_{m_1}=\frac{\alpha T}{Np^\beta}+\frac{\alpha T}{Np^\beta}\frac{(n-2)}{2}(2N-n+1)+\frac{\alpha v}{Np^\beta}(N-n+1)} \quad \text{....................} \quad \mathbf{16}$$

Since the manufacturer willing to wait for backorders of new items during stock out, the demand is assumed to be $d_1(t, p)$

$$Q_{m_2}=\int_v^T d_1(t,p)\,Q(T-t)\,dt$$

$$\mathbf{Q_{m_2}=\frac{\alpha}{p^\beta T}\left[\frac{T^2}{2}-\frac{v^2}{2}\right]} \quad \text{...} \quad \mathbf{17}$$

The production quantity at each replenishment cycle is

$$\mathbf{Q_m = I_1(0)+Q_{m2}}$$

$$\mathbf{\frac{C_m(n,v)}{C_m} = I_1(0)+Q_{m2}}$$

$$\mathbf{C_m(n,v) = (I_1(0)+Q_{m2}).C_m} \quad \text{.......... } \mathbf{18}$$

The lost sale amount is

$$= \int_v^T d_1(t,p)\,[1-(T-t)].dt$$

The cost of lost sale amount

$$\mathbf{L_m(n,v) = \frac{\alpha}{p^{\beta}}\left(\frac{T}{2}+\frac{v^2}{2T}-v\right).r} \quad \text{.......... } \mathbf{19}$$

and $\mathbf{B_m(n,v) = nv}$ **20**

$$\text{Shortage cost} = C_s\int_v^T I_n(t)\,dt$$

$$= C_s\frac{\alpha w(1)}{p^{\beta}}\left(vT-\frac{T^2}{2}-v^2+\frac{v^2}{2}\right)$$

$$\text{Shortage cost} = \mathbf{C_s.\frac{\alpha}{p^{\beta}}\left(vT-\frac{T^2}{2}-\frac{v^2}{2}\right)} \quad \text{.......... } \mathbf{21}$$

Now handling cost

$$\mathbf{H_m(n,v) = H_1^T+\sum_{j=2}^{n-1}H_j^T+H_n^v} \quad \text{.......... } \mathbf{22}$$

Where: $n\geq 3$

$$H_1^T = h\int_0^T I_1(t).dt$$

Omitting the higher powers of a, b & c

$$\mathbf{H_1^T = h\left(P-\frac{\alpha w(1)}{p^{\beta}}\right)\left[\frac{T^2}{2}-\frac{aT^3}{6}-\frac{bT^4}{12}-\frac{cT^5}{20}\right]} \quad \text{.......... } \mathbf{23}$$

$$H_j^T = h\int_0^T I_j(t).dt$$

$$= h\int_0^T\left(-\frac{\alpha w(j)}{p^{\beta}}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right).e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}}.dt$$

$$+h\sum_{\substack{l=j-1\\ \&\ m=\ell-j}}^{2}\left(-\frac{\alpha w(\ell)}{p^{\beta}}\right)\int_0^T\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{a(mT-t)+\frac{b}{2}(mT^2-t^2)+\frac{c}{3}(mT^3-t^3)}$$

$$+h\int_0^T\left(P-\frac{\alpha w(1)}{p^{\beta}}\right)\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-a[(j-1)T+t]-\frac{b}{2}[(j-1)T^2+t^2]-\frac{c}{3}[(j-1)T^3+t^3]}$$

Solving I term

$$\text{I term} = h\int_0^T\left(-\frac{\alpha w(j)}{p^{\beta}}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right).e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}}.dt$$

$$= h\int_0^T\left(-\frac{\alpha w(j)}{p^{\beta}}\right)\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right)\left(1-at-\frac{bt^2}{2}-\frac{ct^3}{3}+......\right).dt$$

After omitting the higher powers of a, b & c

$$= h\left(-\frac{\alpha w(j)}{p^{\beta}}\right)\left[\frac{t^2}{2} - \frac{a}{2}\frac{t^3}{3} - \frac{b}{3}\frac{t^4}{4} - \frac{c}{4}\frac{t^5}{5}\right]_0^T$$

$$\textbf{I term} = h\left(-\frac{\alpha w(j)}{p^{\beta}}\right)\left(\frac{T^2}{2} - \frac{aT^3}{6} - \frac{bT^4}{12} - \frac{cT^5}{20}\right) \quad \text{......................................} \quad \textbf{24}$$

Solving II term :

$$\text{II term} = h\sum_{\substack{\ell=j-1 \\ \& \ m=\ell-j}}^{2}\left(-\frac{\alpha w(\ell)}{p^{\beta}}\right)\left(T + \frac{aT^2}{2} + \frac{bT^3}{6} + \frac{cT^4}{12}\right)^*$$

$$\int [1 + a(mT - t) + \frac{b}{2}(mT^2 - t^2) + \frac{c}{3}(mT^3 - t^3)].dt$$

$$\text{II term} = h\sum_{\substack{\ell=j-1 \\ \& \ m=\ell-j}}^{2}\left(-\frac{\alpha w(\ell)}{p^{\beta}}\right)\left[T^2 + m\left(aT^3 + \frac{bT^4}{2} + \frac{cT^5}{5}\right)\right]$$

Now solving III term

$$\text{III term} = h\int_0^T \left(P - \frac{\alpha w(1)}{p^{\beta}}\right)\left(T + \frac{aT^2}{2} + \frac{bT^3}{6} + \frac{cT^4}{12}\right).^* \, e^{-a[(j-1)T+t] - \frac{b}{2}[(j-1)T^2+t^2] - \frac{c}{3}[(j-1)T^3+t^3]} .dt$$

$$\textbf{III term} = h\left(P - \frac{\alpha w(1)}{p^{\beta}}\right)\left[T^2 - (j-1)\left(aT^3 + \frac{bT^4}{2} + \frac{cT^5}{3}\right)\right] \quad \text{....................} \quad \textbf{25}$$

Put all these value in Equation 22

$$H_j^T = h\left(-\frac{\alpha w(j)}{p^{\beta}}\right)\left(\frac{T^2}{2} - \frac{aT^3}{6} - \frac{bT^4}{12} - \frac{cT^5}{20}\right)$$
$$+ h\sum_{\substack{\ell=j-1 \\ m=\ell-j}}^{2}\left(-\frac{\alpha w(\ell)}{p^{\beta}}\right)\left[T^2 + m\left(aT^3 + \frac{bT^4}{2} + \frac{cT^5}{5}\right)\right]$$
$$+ h\left(P - \frac{\alpha w(1)}{p^{\beta}}\right)\left[T^2 - (j-1)\left(aT^3 + \frac{bT^4}{2} + \frac{cT^5}{5}\right)\right]$$

Now, $H_n^v = h\int_0^v I_n(t).dt$

Here $I_n(t)$ be the inventory level during n^{th} season:

$$\frac{dI_n(t)}{dt} = -(a + bt + ct^2)I_n(t) - \frac{\alpha w(n)}{p^{\beta}}, \text{ where } 0 \le t \le v$$

with boundary condition: $\mathbf{I_n(v) = 0}$

$$\frac{dI_n(t)}{dt} + (a + bt + ct^2)I_n(t) = -\frac{\alpha w(n)}{p^{\beta}}$$

Solution will be

$$I_n(t).e^{at + \frac{bt^2}{2} + \frac{ct^3}{3}} = \int \left(-\frac{\alpha w(n)}{p^{\beta}}\right).e^{at + \frac{bt^2}{2} + \frac{ct^3}{3}}.dt + C_n$$

$$= \left(-\frac{\alpha w(n)}{p^{\beta}}\right)\left(1 + at + \frac{bt^2}{2} + \frac{ct^3}{3} + ...\right).dt + C_n$$

$$I_n(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \left(-\frac{\alpha w(n)}{p^\beta}\right)\left[t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right]+C_n$$

$$I_n(v).e^{av+\frac{bv^2}{2}+\frac{cv^3}{3}} = \left(-\frac{\alpha w(n)}{p^\beta}\right)\left(v+\frac{av^2}{2}+\frac{bv^3}{6}+\frac{cv^4}{12}\right)+C_n$$

$$C_n = -\left(-\frac{\alpha w(n)}{p^\beta}\right)\left(v+\frac{av^2}{2}+\frac{bv^3}{6}+\frac{cv^4}{12}\right)$$

$$\mathbf{I_n(t)=\left(-\frac{\alpha w(n)}{p^\beta}\right).e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}}\left[(t-v)+\frac{a}{2}(t^2-v^2)+\frac{b}{6}(t^3-v^3)+\frac{c}{12}(t^4-v^4)\right]} \quad .. \qquad 26$$

$$H_n^v = h\int_0^v I_n(t).dt$$

$$\mathbf{H_n^v = h\left(-\frac{\alpha w(n)}{p^\beta}\right)\left[-\frac{v^2}{2}-\frac{av^3}{6}-\frac{bv^4}{12}-\frac{cv^5}{20}\right]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 27$$

Now let Is(t) be the supplier's inventory level at t before the beginning of a cycle

$$\mathbf{\frac{dI_s(t)}{dt} = -(a+bt+ct^2)I_s(t)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 28$$

with boundary condition $I_s(0)= I_1(0)$

$$\mathbf{\frac{dI_s(t)}{dt}+(a+bt+ct^2)I_s(t)= 0}$$

Insert Figure 2 here.

Solution with boundary conditions

$\mathbf{I_s(0) = I_1(0)}$

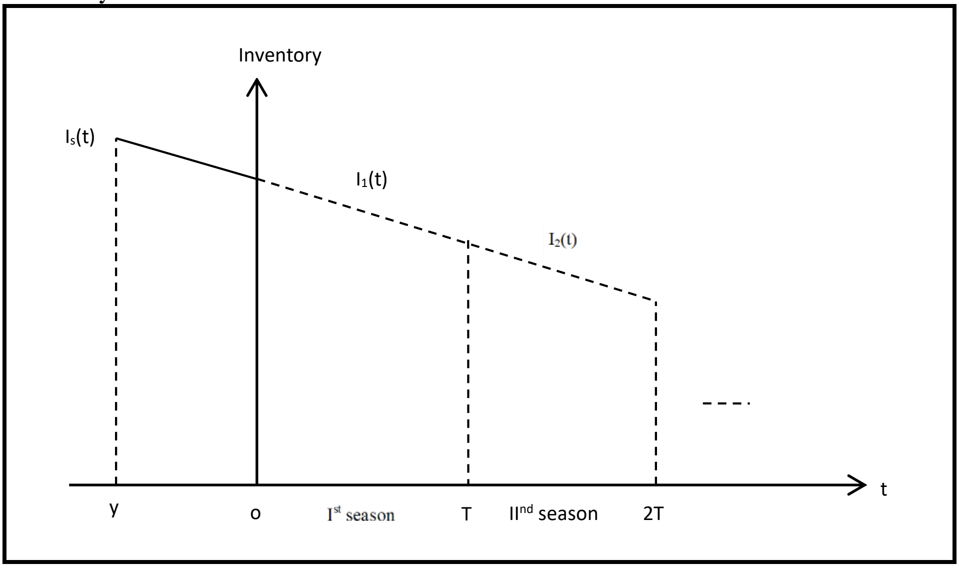
$\mathbf{I_s(0) = C = I_1(0)}$

$$\mathbf{I_s(t) = I_1(0)e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 29$$

Where: $y<t\leq 0$

Figure 2

Inventory Model for Distributor



Now let $I_1(t)$ be the retailer's inventory level at t during Ist season:

$$\frac{dI_1(t)}{dt} = -(a+bt+ct^2)I_1(t) - \frac{\alpha w(1)}{p^\beta} \quad \text{.......} \quad 30$$

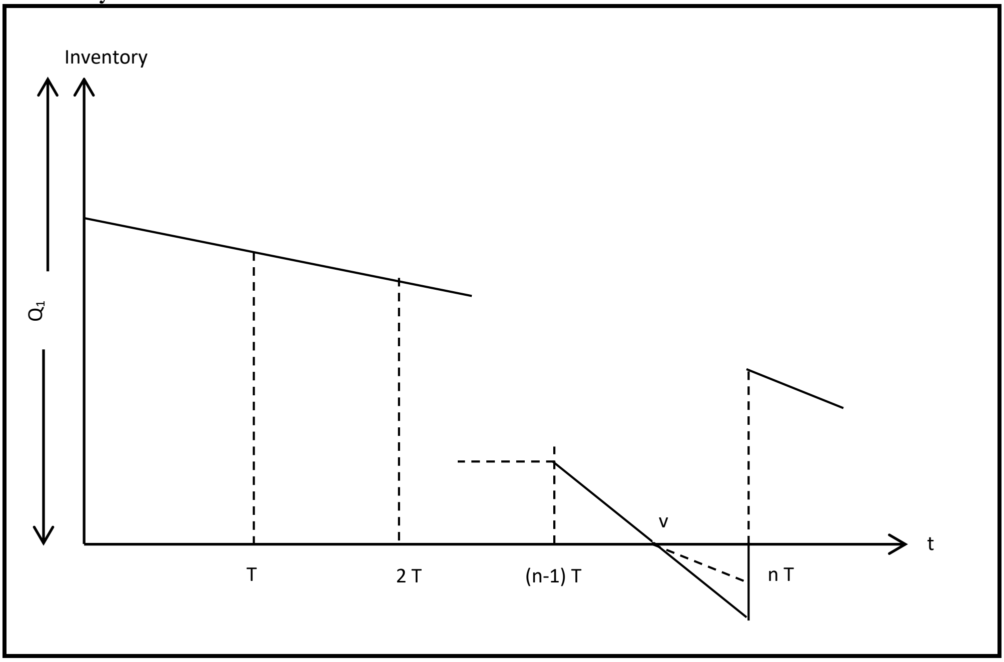
Where: $0 \le t \le T$

with initial condition $I_1(0) = S$

$$\frac{dI_1(t)}{dt} + (a+bt+ct^2)I_t(t) - \frac{\alpha w(1)}{p^\beta}$$

Figure 3

Inventory Model for Retailer



Solution will be

$$I_1(t).e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}} = \int -\frac{\alpha w(1)}{p^\beta}.e^{at+\frac{bt^2}{2}+\frac{ct^3}{3}}.dt + C_1$$

$$= -\frac{\alpha w(1)}{p^\beta}\left[t + \frac{at^2}{2} + \frac{b}{2}\frac{t^3}{3} + \frac{c}{3}\frac{t^4}{4}\right] + C_1 \quad \text{.......} \quad 31$$

Now using initial condition I1(0)= S

$$I_1(0) = -\frac{\alpha w(1)}{p^\beta}.0 + C_1$$

$C_1 = S$

Put this value of C_1 in Equation 29

$$I_1(t) = \left[S - \frac{\alpha w(1)}{p^\beta}\left(t + \frac{at^2}{2} + \frac{bt^3}{6} + \frac{ct^4}{12}\right)\right].e^{-at-\frac{bt^2}{2}-\frac{ct^3}{3}} \quad \text{.......} \quad 32$$

Now if $I_2(t)$ be the inventory level during IInd season:

$$\frac{dI_2(t)}{dt} = -(a+bt+ct^2)I_2(t) - \frac{\alpha w(2)}{p^\beta}, \text{ where } 0 \le t \le T$$

with initial condition $I_2(0) = I_1(T)$

The solution of this equation will be

$$\mathbf{I_2(t)e^{at+\frac{b}{2}t^2+\frac{c}{3}t^3} = -\frac{\alpha w(2)}{p^\beta}\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right)+C_2} \quad \text{.............................} \quad 33$$

$I_2(0) = C_2$

$$\mathbf{C_2 = \left[S-\frac{\alpha w(1)}{p^\beta}\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right)\right]e^{-aT-\frac{b}{2}T^2-\frac{c}{3}T^3}} \quad \text{...........................} \quad 34$$

Put this value of C_2 in Equation 31

$$\mathbf{I_2(t) = \left[S-\frac{\alpha w(1)}{p^\beta}\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right)\right]e^{-a(T+t)+\frac{b}{2}(T^2+t^2)-\frac{c}{3}(T^3+t^3)}}$$

$$\mathbf{-\frac{\alpha w(2)}{p^\beta}\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right).e^{-at-\frac{b}{2}t^2-\frac{c}{3}t^3}} \quad \text{.....................................} \quad 35$$

Now if $I_3(t)$ be the inventory level during 3rd season :

$$\frac{dI_3(t)}{dt} = -(a+bt+ct^2)I_3(t)-\frac{\alpha w(3)}{p^\beta}$$

Solution will be

$$\mathbf{I_3(t)e^{at+\frac{b}{2}t^2+\frac{c}{3}t^3} = -\frac{\alpha w(3)}{p^\beta}\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right)+C_3} \quad \text{.................................} \quad 36$$

using initial condition

$I_3(0) = I_2(T)$

$I_3(0) = C_3$

$$C_3 = \left[S-\frac{\alpha w(1)}{p^\beta}\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right)\right]e^{-2aT-\frac{b}{2}2T^2-\frac{c}{3}2T^3}$$

$$-\frac{\alpha w(2)}{p^\beta}\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right).e^{-aT-\frac{b}{2}T^2-\frac{c}{3}T^3}$$

Put this value of C_3 in Equation 34

$$\mathbf{I_3(t) = \left[S-\frac{\alpha w(1)}{p^\beta}\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right)\right]e^{-a(2T+t)-\frac{b}{2}(2T^2+t^2)-\frac{c}{3}(2T^3+t^3)}}$$

$$\mathbf{-\frac{\alpha w(3)}{p^\beta}\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right).e^{-at-\frac{b}{2}t^2-\frac{c}{3}t^3}}$$

$$\mathbf{-\frac{\alpha w(2)}{p^\beta}\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right)e^{-a(T+t)-\frac{b}{2}(T^2+t^2)-\frac{c}{3}(T^3+t^3)}} \quad \text{...........................} \quad 37$$

If $I_4(t)$ be the inventory level during 4th season

$$\frac{dI_4(t)}{dt} = -(a+bt+ct^2)I_4(t)-\frac{\alpha w(4)}{p^\beta}$$

Solution will be

$$\mathbf{I_4(t).e^{at+\frac{b}{2}t^2+\frac{c}{3}t^3} = -\frac{\alpha w(4)}{p^\beta}\left(t+\frac{at^2}{2}+\frac{bt^3}{6}+\frac{ct^4}{12}\right)+C_4} \quad \text{.................................} \quad 38$$

using initial condition

$I_4(0) = I_3(T)$

$I_4(0) = C_4$

$$\mathbf{C_4 = \left[S-\frac{\alpha w(1)}{p^\beta}\left(T+\frac{aT^2}{2}+\frac{bT^3}{6}+\frac{cT^4}{12}\right)\right]e^{-a.3T-\frac{b}{2}.3T^2-\frac{c}{3}.3T^3}}$$

$$-\frac{\alpha w(3)}{p^{\beta}}\left(T+\frac{aT^{2}}{2}+\frac{bT^{3}}{6}+\frac{cT^{4}}{12}\right).e^{-at-\frac{b}{2}t^{2}-\frac{c}{3}t^{3}}$$

$$-\frac{\alpha w(2)}{p^{\beta}}\left(T+\frac{aT^{2}}{2}+\frac{bT^{3}}{6}+\frac{cT^{4}}{12}\right)e^{-2aT-\frac{b}{2}.2T^{2}-\frac{c}{3}T^{3}} \quad \text{..........} \quad 39$$

Put this value of C_4 in Equation 36

$$I_4(t)=-\frac{\alpha w(4)}{p^{\beta}}\left(t+\frac{at^{2}}{2}+\frac{bt^{3}}{6}+\frac{ct^{4}}{12}\right).e^{-at-\frac{b}{2}t^{2}-\frac{c}{3}t^{3}}$$

$$-\frac{\alpha w(3)}{p^{\beta}}\left(T+\frac{aT^{2}}{2}+\frac{bT^{3}}{6}+\frac{cT^{4}}{12}\right)e^{-a(T+t)-\frac{b}{2}(T^{2}+t^{2})-\frac{c}{3}(T^{3}+t^{3})}$$

$$-\frac{\alpha w(2)}{p^{\beta}}\left(T+\frac{aT^{2}}{2}+\frac{bT^{3}}{6}+\frac{cT^{4}}{12}\right)e^{-a(2T+t)-\frac{b}{2}(2T^{2}+t^{2})-\frac{c}{3}(2T^{3}+t^{3})}$$

$$+\left[S-\frac{\alpha w(1)}{p^{\beta}}\left(T+\frac{aT^{2}}{2}+\frac{bT^{3}}{6}+\frac{cT^{4}}{12}\right)\right]e^{-a(3T+t)-\frac{b}{2}(3T^{2}+t^{2})-\frac{c}{3}(3T^{3}+t^{3})} \quad \text{..........} \quad 40$$

Similarly

$$I_j(t)=-\frac{\alpha w(j)}{p^{\beta}}\left(t+\frac{at^{2}}{2}+\frac{bt^{3}}{6}+\frac{ct^{4}}{12}\right).e^{-at-\frac{b}{2}t^{2}-\frac{c}{3}t^{3}}$$

$$-\sum_{\substack{\ell=j-1\\ m=\ell-j}}^{2}\frac{\alpha w(\ell)}{p^{\beta}}\left(T+\frac{aT^{2}}{2}+\frac{bT^{3}}{6}+\frac{cT^{4}}{12}\right).e^{a(mT-t)+\frac{b}{2}(mT^{2}-t^{2})+\frac{c}{3}(mT^{3}-t^{3})}$$

$$+\left[S-\frac{\alpha w(1)}{p^{\beta}}\left(T+\frac{aT^{2}}{2}+\frac{bT^{3}}{6}+\frac{cT^{4}}{12}\right)\right]e^{-a[(j-1)T+t]-\frac{b}{2}[(j-1)T^{2}+t^{2}]-\frac{c}{3}[(j-1)T^{3}+t^{3}]}$$

Now the replenishment cycle and shortage length are set at nT and (T-v) units of time respectively.

Now when $y\leq 0$, it means supplier's delivery is completed early, the retailer's unit time profit without late delivery is :

$F_R(n, v)$ = $\frac{1}{nT}$ [Sales Revenue-Purchasing Cost-Lost Sale Cost-Processing Cost -Inventory Holding Cost-Ordering Cost-Shortage Cost]

$$F_R(n,v)=\frac{1}{nT}[R_R(n,v)-C_R(n,v)-L_R(n,v)-B_R(n,v)-H_R(n,v)-C_{RO}-S_{CR}]\text{...} \quad 41$$

Where: $N\leq N$, $0<v\leq T$

In this case the supplier suffers inventory holding cost until the target due to early delivery, the supplier's unit time profit with early delivery by y unit time $F_S(n, v, \xi, y)$ is

$$F_S(n,v,\xi,y)=\frac{1}{nT}\left[(I_1(0)+Q_2)(C-C_p)[I_s(y)-I_1(0)]C_P-h\int_y^0 I_s(t)dt-\xi\right]\text{.....} \quad 42$$

Where: $n<N$, $0<v<T$, $y\leq 0$ and C_P = purchasing cost per unit of supplier

Now for the retailer:

$$R_R(n,v)=pQ_{R1}+\lambda pQ_{R2} \quad \text{..........} \quad 43$$

$$Q_{R1}=\int_0^T d_1(t,p)dt+\sum_{j=2}^{n-1}\int_0^T d_j(t,p)dt+\int_0^v d_n(t,p)dt$$

$$Q_{R1}=\frac{\alpha T}{p^{\beta}}+\frac{\alpha T}{Np^{\beta}}\frac{(n-2)}{2}(2N-n+1)+\frac{\alpha v}{Np^{\beta}}(N-n+1) \quad \text{..........} \quad 44$$

Since the retailer is willing to wait for backorders of new items during stock out, the demand is assumed to be $d_1(t, p)$:

$$Q_{R2} = \int_v^T d_1(d,p)\theta(T-t)dt$$

$$\mathbf{Q_{R2} = \frac{\alpha}{p^{\beta}T}\left(\frac{T^2}{2} - \frac{v^2}{2}\right)} \quad \text{........} \quad \mathbf{45}$$

The ordered quantity at each replenishment cycle is

$\mathbf{Q_R = I_1(0) + Q_{R2}}$

$\mathbf{\frac{Q_R(n, v)}{C_R} = I_1(0) + Q_{R2}}$

$$\mathbf{Q_R(n, v) = [I_1(0) + Q_{R2}].C_R} \quad \text{........} \quad \mathbf{46}$$

The lost sale amount= $\int_v^T d_1(d,p)[1-\theta(T-t)]dt$

$$= \frac{\alpha}{p^{\beta}}\left(\frac{T}{2} + \frac{v^2}{2T} - v\right)$$

The cost of lost sale amount:

$$\mathbf{L_R(n,v) = \frac{\alpha}{p^{\beta}}\left(\frac{T}{2} + \frac{v^2}{2T} - v\right).r} \quad \text{........} \quad \mathbf{47}$$

and processing cost:

$$\mathbf{B_R(n, v) = nv} \quad \text{........} \quad \mathbf{48}$$

Handling cost

$$H_R(n,v) = H_1^T + \sum_{j=2}^{n-1} H_j^T + H_n^v, \text{ where } n \geq 3$$

$$H_1^T = h\int_0^T I_1(t).dt$$

$$\mathbf{H_1^T = h\left[S\left(T - \frac{aT^2}{2} - \frac{bT^3}{6} - \frac{cT^4}{12}\right) - \frac{\alpha}{p^{\beta}}\left(\frac{T^2}{2} - \frac{aT^2}{6} - \frac{bT^4}{12} - \frac{cT^5}{20}\right)\right]} \quad \text{........} \quad \mathbf{49}$$

Now, $H_j^T = h\int_0^T I_j(t).dt$

Solving and Omitting the higher powers of a, b & c

$$\mathbf{H_j^T = -\frac{h\alpha w(j)}{p^{\beta}}\left(\frac{T^2}{2} - \frac{aT^2}{6} - \frac{bT^4}{12} - \frac{cT^5}{20}\right)}$$
$$\mathbf{- \sum_{\substack{\ell=j-1 \\ m=\ell-j}}^{2} \frac{h\alpha w(\ell)}{p^{\beta}}\left(T^2 + amT^3 + \frac{b}{2}mT^4 + \frac{c}{3}mT^5\right)}$$
$$\mathbf{+ h\left[S\left\{T - \frac{aT^2}{2} - \frac{bT^3}{6} - \frac{cT^4}{12} - (j-1)\left(aT^2 + \frac{b}{2}T^3 + \frac{c}{3}T^4\right)\right\}\right.}$$
$$\mathbf{\left. - \frac{\alpha}{p^{\beta}}\left\{T^2 - (j-1)\left(aT^3 + \frac{b}{2}T^4 + \frac{c}{3}T^5\right)\right\}\right]} \quad \text{........} \quad \mathbf{50}$$

$$H_n^v = \int_0^v I_n(t).dt$$

at first we will calculate $I_n(t)$

$$\frac{dI_n(t)}{dt} = -\left(a + bt + ct^2\right)I_n(t) - \frac{\alpha w(n)}{p^\beta},$$

Where: $0 \le t \le T$

with boundary condition $I_n(v) = 0$

The solution of this equation will be

$$I_n(t) = \frac{\alpha w(n)}{p^\beta} e^{-at - \frac{b}{2}t^2 - \frac{c}{3}t^3} \left[(v-t) + \frac{a}{2}(v^2 - t^2) + \frac{b}{6}(v^3 - t^3) + \frac{c}{12}(v^4 - t^4)\right]$$

Now,

$$H_n^v = h\int_0^v \frac{\alpha w(n)}{p^\beta}\left(1 - at - \frac{b}{2}t^2 - \frac{c}{3}t^3 ....\right)\left[(v-t) + \frac{a}{2}(v^2 - t^2) + \frac{b}{6}(v^3 - t^3) + \frac{c}{12}(v^4 - t^4)\right].dt$$

$$= \frac{h\alpha w(n)}{p^\beta}\left[v^2 - \frac{v^2}{2} + \frac{a}{2}\left(v^3 - \frac{v^3}{3}\right) + \frac{b}{6}\left(v^4 - \frac{v^4}{4}\right) + \frac{c}{12}\left(v^5 - \frac{v^5}{5}\right)\right.$$

$$\left. - \frac{av^3}{2} + \frac{av^3}{3} - \frac{bv^4}{6} + \frac{bv^4}{8} - \frac{cv^5}{12} + \frac{cv^5}{15}\right]$$

$$= \frac{h\alpha w(n)}{p^\beta}\left[\frac{v^2}{2} + \frac{av^3}{3} + \frac{bv^4}{8} + \frac{cv^5}{15} - \frac{av^3}{6} - \frac{bv^4}{24} - \frac{cv^5}{60}\right]$$

$$\mathbf{H_n^v = \frac{h\alpha w(n)}{p^\beta}\left[\frac{v^2}{2} + \frac{av^3}{6} + \frac{bv^4}{12} + \frac{cv^5}{20}\right]} \quad \text{...} \quad \mathbf{51}$$

Shortage cost (S. C.) $= C_s\int_v^T d_1(t,p)dt$

$$= C_s \frac{\alpha w(1)}{p^\beta}(T - v)$$

$$\mathbf{S.C. = C_s \frac{\alpha}{p^\beta}(T - v)} \quad \text{..} \quad \mathbf{52}$$

Now when the supplier's lead time y>0.

The retailer's unit time profit with late delivery by y unit time $F_{Rd}(n, v, y)$ is:

$$\mathbf{F_{RdR}(n, v, y) = \frac{1}{nT}[Rd_R(n, v, y) - Cd_R(n, v, y) - Ld_R(n, v, y)}$$
$$\mathbf{- B_R(n, v) - Hd_R(n, v, y) - C_{OR}]} \quad \text{...} \quad \mathbf{53}$$

Where: $0 < v \le T$, $y > 0$ and $n \le N$

Here $\quad R_{d_R}(n, v, y) = pQ_{R1d} + \lambda pQ_{R2}$

We have assumed that the retailer cannot promise the back-ordered quantity for the customers due to the supplier's delay. Therefore, the shortage without backordering will occur during the time interval [0, y].

$$Q_{R1d} = \sum_{j=1}^{n-1}\int_0^T d_j(t,p)dt + \int_0^v d_n(t,p)dt - \int_0^y d_1(t,p)dt$$

$$= \sum_{j=1}^{n-1}\int_0^T \frac{\alpha w(j)}{p^\beta}.dt + \int_0^v \frac{\alpha w(n)}{p^\beta}.dt - \int_0^y \frac{\alpha w(1)}{p^\beta}.dt$$

$$= \sum_{j=1}^{n-1}\frac{\alpha}{p^\beta}\frac{N - j + 1}{N}.T + \frac{\alpha\left(N - n + 1\right)}{Np^\beta}.v - \frac{\alpha}{p^\beta}.y$$

$$=\frac{\alpha T}{Np^{\beta}}\left[(n-1)(N+1)-(1+2+.....n-1)\right]+\frac{\alpha}{p^{\beta}}\frac{(N-n+1)v}{N}-\frac{\alpha}{p^{\beta}}.y$$

$$=\frac{\alpha T}{Np^{\beta}}\left[(n-1)(N+1)-\frac{n-1}{2}(2+n-2)\right]+\frac{\alpha}{p^{\beta}}\frac{(N-n+1)v}{N}-\frac{\alpha}{p^{\beta}}.y$$

$$\mathbf{Q_{R1d}=\frac{\alpha T}{Np^{\beta}}\frac{(n-1)}{2}(2N-n+2)+\frac{\alpha}{p^{\beta}}\frac{(N-n+1)v}{N}-\frac{\alpha}{p^{\beta}}.y} \quad \text{.........................} \quad 54$$

The order quantity at each replenishment is:

$\mathbf{Q_{Rd} = I_1(y) + Q_{R2}}$

$$\frac{Cd(n,v,y)}{C}=I_1(y)+Q_{R2}$$

Cd (n, v, y) = [I_1(y) + Q_{R2}].C **55**

The lost sale amount is

$$\int_0^y d_1(t,p)dt+\int_v^T d_1(t,p)[1-\theta(T-t)]dt$$

$$=\frac{\alpha w(1)}{p^{\beta}}.y+\frac{\alpha w(1)}{p^{\beta}}\int_v^T\left(1-1+\frac{T-t}{T}\right).dt$$

$$=\frac{\alpha}{p^{\beta}}.y+\frac{\alpha}{p^{\beta}}\left[t-\frac{t^2}{2T}\right]_v^T$$

$$=\frac{\alpha}{p^{\beta}}.y+\frac{\alpha}{p^{\beta}}\left(\frac{T}{2}-v+\frac{v^2}{2T}\right)$$

The cost of lost sale amount is:

$$\mathbf{Ld(n,\ v,y)=\frac{\alpha y}{p^{\beta}}.r+\frac{\alpha}{p^{\beta}}\left(\frac{T}{2}-v+\frac{v^2}{2T}\right).r} \quad \text{...} \quad 56$$

and **B(n, v)= nv**... **57**

Now, $\mathbf{Hd(n,v,y)=H_1^T+\sum_{j=2}^{n-1}H_j^T+H_n^v-h\int_0^v I_1(t).dt}$ **58**

Since there is a potential lose F_R(n, v, y) - F_{Rd} (n,v,y) on the retailer due to the supplier's delay, the supplier has to compensate the retailer. Therefore, the retailer's unit time profit is : -

F_{Rd}(n, v, y)+[F_R(n, v)-F_{Rd}(n, v, y)]= F_R(n, v) **59**

i.e. the retailer maintains his profit regardless of the supplier's delivery behavior. The supplier's unit time profit with late delivery by y unit time F_{sd}(n, v, ξ , y) is:

F_{sd}(n, v, ξ, y)= $\frac{1}{nT}$ (Sales Revenue-Deteriorated Cost-Compensated Cost-Managing Cost) **60**

Where: n≤N, 0<v≤T and y>0

from equation 40 and 56 the supplier's expected unit time profit E_s(n, v, ξ) is:

$$\mathbf{E_s(n,v,\xi)=\int_{-\infty}^{0}F_s(n,v,\xi,y)\ f(y)dy+\int_0^{\infty}F_{sd}(n,v,\xi,y)\ f(y)dy} \quad 61$$

If the retailer decides the order quantity independently then the problem can be formulated as:

Max: F_R(n, v) **62**

Where: S.t.= 1≤n≤N, and 0<v≤T

Now when supplier's capital constraint is considered when the supplier's lead time $y \leq 0$, the supplier's cost is:

G(n, v, ξ, y)= [Production Cost+Inventory Holding Cost+ Deteriorated Cost+managing cost] 63

When the supplier's lead time $y > 0$, the supplier's cost is

Gd(n, v, ξ, y)= [Production Cost+Compensated Cost+ Deteriorated Cost+Managing Cost] 64

From these equations, the supplier's expected cost is

$$E_G(n, v, \xi) = \int_{-\infty}^{0} G(n, v, \xi, y)\, f(y)dy + \int_{0}^{\infty} Gd(n, v, \xi, y)\, f(y)dy \quad \text{.................. } 65$$

Where: $n \leq N$, $0 < v \leq T$ and $\xi \geq 0$

The supplier's optimization problem can be formulated as

Max: $E_s(n^*_R, v^*_R(n_R), \xi)$ **66**

S.t.: $EG(n^*_R, v^*_R(n_R), \xi) \leq W$

Thus supplier can calculate its optimal inventory level using the equation 66 to maximize its factor.

IV. OBSERVATIONS AND RESULTS

The model can be solved using mathematical packages like mat lab. considering the various scenario, the time period for inventory is always favoring the permissible delay in payments. We can say that this study aims to derive the retailer's optimal replenishment cycle, shortage period, economic order quantity, and the managing cost of the supplier to maximize the expected unit time profit. In this study, we have investigated a supply chain inventory model for deteriorating items in which demand depends upon the time and price. The shortages allowed and partially backlogged, and it is assumed that backlogging rate varies inversely as the waiting time. Demand is seasonal, along with the expiration rate. The decision variables in this article are the length of the replenishment cycle, shortage period, order quantity, and the managing cost.

V. CONCLUSION

One of the significant logistics tasks is inventory management, although it has various other tasks related to inventory. But has overall responsibility for the movement and storage- of materials; it is undoubtedly impossible to separate inventory management from other decisions about the supply chain.

In this study, we develop a perishable inventory model with time-dependent demand, including the condition of shortage. The product will deteriorate with time, decreasing utility or price from the original. Moreover, customers' demand declines when the product is near its expiration date. The rate of deterioration of an item is either constant, time-dependent, or stock-dependent. For example, some things made of glass, china clay, or ceramic break during their storage period, for which the deterioration rate depends upon the total inventory size. In addition, decaying items such as photographic films, electronic goods, fruits, and vegetables slowly lose efficacy with time. In the existing models, it is usually assumed that the deteriorated units are a complete loss to the inventory management.

A vital aspect of our model is incorporating the product's expiration date, which adds another layer of complexity to inventory management. By recognizing the finite shelf life of perishable goods, participants in the supply chain can make more informed decisions regarding stock levels, maximizing profitability across the entire supply chain.

To optimize profits, it is imperative that all stakeholders—manufacturers, suppliers, and retailers—collaborate closely. The study emphasizes that effective communication and timely information sharing among supply chain partners are fundamental to achieving the desired outcomes. This collaborative approach fosters a more agile supply chain capable of responding to demand and supply conditions changes.

Furthermore, the model's convexity has been rigorously verified, supported by numerical examples illustrating its practical applicability in real-world scenarios. These illustrations demonstrate how the proposed models can lead to enhanced efficiency and profitability, thus validating their potential to transform the management of perishable product supply chains.

This study can be converted for linear deterioration rate by taking c= 0 and constant deterioration rate by taking b and c equal to zero. Even a mathematical programming model is proposed for retailers to maximize their profit.

REFERENCES

Aggarwal, S. P., & Jaggi, C. K. (1995). Ordering policies of deteriorating items under permissible delay in payments. *Journal of the Operational Research Society*, *46*(5), 658-662. https://doi.org/10.2307/2584538

Barbosa, L. C., & Friedman, M. (1978). Deterministic inventory lot size models—a general root law. *Management Science*, *24*(8), 819-826.

Chang, H. J., & Dye, C. Y. (1999). An EOQ model for deteriorating items with time varying demand and partial backlogging. *Journal of the Operational Research Society*, *50*(11), 1176-1182.

Dash, B. P., Singh, T., & Pattnayak, H. (2014, January). An inventory model for deteriorating items with exponential declining demand and time-varying holding cost. *American Journal of Operations Research*, *4*(1), 1-7. Doi: 10.4236/ajor.2014.41001.

Dye, C. Y., Ouyang, L. Y., & Hsieh, T. P. (2007). Deterministic inventory model for deteriorating items with capacity constraint and time-proportional backlogging rate. *European Journal of Operational Research*, *178*(3), 789-807.

Feng, H., Li, J., & Zhao, D. (2013). Retailer's optimal replenishment and payment policies in the EPQ model under cash discount and two-level trade credit policy. *Applied Mathematical Modelling*, *37*(5), 3322-3339.

Ghare, P. M., & Schrader, G. F. (1963). A model for an exponentially decaying inventory. *Journal of Industrial Enginering*, *14*, 238-243.

Goyal, S. K. (1985, April). Economic order quantity under conditions of permissible delay in payments. *The Journal of the Operational Research Society*, *36*(4), 335-338. https://doi.org/10.2307/2582421.

Hsu, P. -H., Wee, H. M., & Teng, H. -M. (2006). Optimal lot sizing for deteriorating items with expiration date. *Journal of Information & Optimization Sciences*, *27*(2), 271-286. https://doi.org/10.1080/02522667.2006.10699692.

Hwang, H., & Shinn, S. W. (1997, June). Retailer's pricing and lot sizing policy for exponentially deteriorating products under the condition of permissible delay in payments. *Computers & Operations Research*, *24*(6), 539-547.

Jamal, A. M. M., Sarker, B. R., & Wang, S. (1997, August). An ordering policy for deteriorating items with allowable shortage and permissible delay in payment. *Journal of the Operational Research Society*, *48*(8), 826-833.

Liao, J. -J., Chung, K. -J., & Huang, K. -N. (2013). A deterministic inventory model for deteriorating items with two warehouses and trade credit in a supply chain system. *International Journal of Production Economics*, *146*(2), 557-565. https://doi.org/10.1016/j.ijpe.2013.08.001

Mahata, G. C. (2012, February). An EPQ-based inventory model for exponentially deteriorating items under retailer partial trade credit policy in supply chain. *Expert Systems with Applications*, *39*(3), 3537-3550. https://doi.org/10.1016/j.eswa.2011.09.044

Mandal, B. A., & Phaujdar, S. (1989, May). An inventory model for deteriorating items and stock-dependent consumption rate. *Journal of the Operational Research Society*, *40*(5), 483-488.

Min, J., Zhou, Y. -W., & Zhao, J. (2010, November). An inventory model for deteriorating items under stock-dependent demand and two-level trade credit. *Applied Mathematical Modelling*, *34*(11), 3273-3285.

Petruzzi, N. C., & Dada, M. (1999). Pricing and the newsvendor problem: A review with extensions. *Operations Research*, *47*(2), 183-194.

Saha, S., & Sen, N. (2019). An inventory model for deteriorating items with time and price dependent demand and shortages under the effect of inflation. *International Journal of Mathematics in Operational Research, 14*(3), 377-388.

San-José, L. A., Sicilia, J., & Abdul-Jalbar, B. (2021). Optimal policy for an inventory system with demand dependent on price, time and frequency of advertisement. *Computers & Operations Research*, *128*, 105169.

Shah, N. H., & Shukla, K. T. (2009). Deteriorating inventory model for waiting time partial backlogging. *Applied Mathematical Sciences*, *3*(9), 421-428.

Shinn, S. W., & Hwang, H. (2003). Optimal pricing and ordering policies for retailers under order-size-dependent delay in payments. *Computers & Operations Research*, *30*(1), 35-50.

Singh, S. (2017). Optimal ordering policy for deteriorating items having constant demand and deterioration rate. *Theoretical Economics Letters*, *7*(7), 2152-2158.

Wu, K. S., Ouyang, L. Y., & Yang, C. T. (2006). An optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging. *International Journal of Production Economics*, *101*(2), 369-384.